Applied Econometrics

Lecture 9: Production Functions

- This lecture continues the macro theme and looks at one of the most important aspects of applied work in this area production functions. It will:
  - Provide an overview of theoretical models and the problems involved in applying them
  - Discuss how they have been applied
  - Introduce you to the applied exercise in which you estimate an aggregate production function for South Africa and consider how to develop the analysis
- This and the referenced reading should provide you with an understanding of the issue and problems of estimating production functions and a hands on understanding of how to undertake such analyses.

Production Functions

Productions functions are an important component of applied economic analysis:

- **Macro level**: combined with MP theory to explain prices of factors of production and the extent to which utilised. Important for growth/distribution
- **Micro level**: used to investigate substitutability between factors returns to scale
- **Both**: used to consider what proportion of growth is the result of increase in factor inputs, returns to scale and technical progress

Have been object of considerable controversy -Cambridge capital controversy - but remain an important component of applied economics and with the resurgence of growth theory in the form of new growth theory are increasingly important.

Basic Model:

$$Q = Q(K, L)$$

Defines the maximum output Q given inputs of capital K and labour L. Note these are flow variables and we assume variable/divisible and continuously substitutable. Technical questions of how to get from K and to the best Q is assumed solved, but substitution means can get a given from a number of combinations and so need to consider the minimum cost combination

$$\frac{\partial Q}{\partial L} = MP_L$$
$$\frac{\partial Q}{\partial K} = MP_K$$

which are positive but diminishing marginal productivity: diminishing returns to a factor

$$Q(K,L) = (\lambda K, \lambda L) = \lambda^n Q(k,L)$$

homogeneous of degree n,
if $n < 1$ decreasing returns,
$n = 1$ constant returns
$n > 1$ increasing returns.

Questions of what determines the proportions by which the factors are combined is an economic one.

At micro level use model of firm behaviour maximising profits π

$$\pi = pQ - mK - wL$$
$$\text{st } Q = Q(K,L)$$

Assuming perfect competition in production and factor markets the factor prices p, m, w are given and the Lagrangian

$$L = pQ - mK - wL - \lambda[Q - Q(K,L)]$$

An alternative economic model is to assume Q is predetermined and minimise costs subject to the level of output.

$$\min C = mK + wL$$
$$\text{st } Q^0 = Q(K,L)$$
$$L = mK + wL - \lambda[Q^0 - Q(K,L)$$

Both the profit maximizing and cost minimising models imply that factors are combined so as to equate the MRS with the ratio of factor prices.

$$MRS = \frac{\partial Q}{\partial L} / \frac{\partial Q}{\partial K} = \frac{w}{m}$$

Even if the assumption of perfect competition is dropped profit maximising still implies this.

MRS measures the extent it is possible to substitute one factor for another in the production of a given output, but its size will depend on the units of measurement of L and K. For this reason the elasticity of substitution is used

$$\sigma = \frac{d(K/L)}{K/L} / \frac{dMRS}{MRS}$$

The higher σ the more the substitution possibilities.

Cobb-Douglas Production Function

In applied work the most commonly used form of the production function is the Cobb Douglas. This resulted from Douglas observing that the share of national output going to labour was approximately constant over time

$$wL = \beta pQ$$

An underlying production function that would give rise to this observation is of the form

$$Q = AK^{\alpha}L^{\beta}$$

This has a number of convenient properties α, β are the elasticities of output and A can be considered an efficiency parameter

$$\frac{\partial Q}{\partial K} = \alpha AK^{\alpha-1}L^{\beta} = \alpha\frac{Q}{K}$$

$$\frac{\partial Q}{\partial L} = \alpha AK^{\alpha}L^{\beta-1} = \beta\frac{Q}{L}$$

Assuming the firm is a price taker and profit maximiser then

$$\frac{\partial Q}{\partial K} = \alpha\frac{Q}{K} = \frac{m}{p}$$

$$\frac{\partial Q}{\partial L} = \beta\frac{Q}{L} = \frac{w}{p}$$

which can be written as:

$$\alpha = \frac{mK}{pQ}$$

$$\beta = \frac{wL}{pQ}$$

which is the regularity Douglas observed. So if the MP conditions hold then α and β in the C-D are equal to the respective shares of capital and labour in the share of national output (also need constant returns to scale). As before this result can be derived from a cost minimising approach and for both models the optimising conditions imply:

$$\frac{K}{L} = \left(\frac{\alpha}{\beta}\right)\left(\frac{w}{m}\right)$$

For any given factor price ratio the greater is α/β the greater is K/L. So size of α relative to β determines capital intensity of the production process represented by the C-D

C-D is also homogeneous of degree $\alpha + \beta$

$$Q(\lambda K, \lambda L) = A(\lambda K)^{\alpha}(\lambda L)^{\beta} = \lambda^{\alpha+\beta}AK^{\alpha}L = \lambda^{\alpha+\beta}q(K,L)$$

$\alpha + \beta > 1$ increasing returns to scale
$\alpha + \beta = 1$ constant returns to scale
$\alpha + \beta < 1$ decreasing returns to scale
NB the returns to scale property is the same at all levels of output and the C-D implies a

constant elasticity of substitution that is equal to one

$$\sigma = \frac{d(K/L)}{K/L} / \frac{d(w/m)}{w/m}$$

There is a problem that for the first order solutions to be unique decreasing return to scale are required, but is unlikely and if it doesn't hold can get some strange results. The answer is to relax the assumptions of perfect competition, where prices are exogenously given, but this make prices endogenous and suggests product demand and factor supply equations should be added to the system.

Note that we can also use the framework to look at factor demands and productivity. Consider:

$$Y = AK^{\alpha}L^{\beta}$$

to operationalise this we normally take logs and estimate

$$\log Y = \log A + \alpha \log K + \beta \log L$$

using lower case for logs we can rewrite as

$$y = \alpha_0 + \alpha_1 k + \beta l$$

assuming constant returns to scale $\alpha_1 + \beta = 1$ so

$$y = \alpha_0 + \alpha_1 k + (1 - \alpha_1)l$$

$$y = \alpha_0 + \alpha_1 k + l - \alpha_1 l$$

so

$$y - l = \alpha_0 + \alpha_1 k - \alpha_1 l$$

$$y - l = \alpha_0 + \alpha_1 (k - l)$$

this is sometimes used to overcome problems of multicollinearity. If we don't assume constant returns to scale then:

$$y - l = \alpha_0 + \alpha_1 k + \beta l - l$$

$$y - l = \alpha_0 + \alpha_1 k + (\beta - 1)l$$

which can be reparametarised fro estimation as

$$y - l = \alpha_0 + \alpha_1 k + \alpha_2 l$$

CES Production Function

A more general, though also more complex production function is one with constant but

not necessarily unity elasticity of substitution, the CES production function. Arrow, Chenery,Minhas and Solow (1961) estimated cross section equations

$$\frac{Q}{L} = \frac{1}{\beta}\left(\frac{w}{p}\right)^{\chi}$$

now a C-D with profit maximising and perfect competition would imply $\chi = 1$, but they consistently found it to be less, implying that the underlying function must be:

$$Q = \gamma[\delta K^{-\theta} + (1-\delta)L^{-\theta}]^{-1/\theta}$$

where γ is efficiency parameter similar to A in the C-D. This means

$$\frac{\delta Q}{\delta K} = \frac{\delta\gamma}{K^{1+\theta}}[\delta K^{-\theta} + (1-\delta)L^{-\theta}]^{-\frac{(1+\theta)}{\theta}} = \frac{\delta}{\gamma^{\theta}}\left(\frac{\theta}{K}\right)^{1+\theta}$$

$$\frac{\delta Q}{\delta K} = \frac{(1-\delta)\gamma}{K^{1+\theta}}[\delta K^{-\theta} + (1-\delta)L^{-\theta}]^{-\frac{(1+\theta)}{\theta}} = \frac{(1-\delta)}{\gamma^{\theta}}\left(\frac{\theta}{K}\right)^{1+\theta}$$

Marginal productivity equations under profit maximization and perfect competition are:

$$\frac{\delta}{\gamma^{\theta}}\left(\frac{\theta}{K}\right)^{1+\theta} = \frac{m}{p}$$

$$\frac{(1-\delta)}{\gamma^{\theta}}\left(\frac{\theta}{K}\right)^{1+\theta} = \frac{w}{p}$$

which leads to the SMAC estimating equation:

$$\frac{Q}{L} = \frac{1}{\beta}\left(\frac{w}{p}\right)^{\chi} \text{ with } \chi = \frac{1}{1+\theta} \text{ and } \frac{1}{\beta} = \left(\frac{\gamma^{\theta}}{1-\delta}\right)$$

MRS is

$$\left(\frac{1-\delta}{\delta}\right)\left(\frac{K}{L}\right)^{1+\theta}$$

and elasticity of substitution is:

$$\sigma = \frac{1}{1+\theta}$$

θ is known as substitution parameter as $\theta = \frac{1}{\sigma} - 1$

- when $\theta = \infty$ then $\sigma = 0$ and substitution is impossible
- when $\theta = -1$ then $\sigma = \infty$ and isoquants are straight lines
- when $\theta = 0$ then $\sigma = 1$ and we have the C-D production function

Can also write

$$\frac{wL}{mK} = \left(\frac{1-\delta}{\delta}\right)\left(\frac{K}{L}\right)^{\theta}$$

So for a given K/L and θ, δ will determine the shares of capital and labour. It is the ”distribution parameter”. For the C-D the ratio of factor shares was constant.

Note that the CES above implies constant returns to scale, but we can generalise to

$$Q = \gamma[\delta K^{-\theta} + (1-\delta)L^{-\theta}]^{-v/\theta}$$

v is then the returns to scale factor:

- $v > 1$ increasing returns to scale
- $v = 1$ constant returns to scale
- $v > 1$ decreasing returns to scale

Applying the models

Serious conceptual problems with aggregate production functions:

- Capital controversy: theoretical inconsistencies in N-C production function and distribution.
- Individual firms unlikely to have the same production function. C-D is in logs so means should be geometric rather than arithmetic.
- Production function is only one equation of a simultaneous system: marginal productivity conditions, need to be aggregated as well.
- Existence of external economies of scale: whole > sum of parts
- Firms/industries can have widely different outputs and techniques of production: α and β unlikely to be distributed independently of K and L. Can be affected by market and industry structure so cannot be a purely technical relationship.

Serious practical problems

- Aggregating labour inputs:
  - In practice aggregate monetary value of inputs, deflated by a labour input price index, or use unweighted measures of flows (total person hours) or stock (total no. employees).
  - types of labour skills/quality
  - weighting
  - changes in quality
- Aggregating output:
  - weights: usually use market prices
  - changes in quality
  - Use of gross output and value added measures
- Aggregating capital inputs causes greatest problems:
  - Often use value of the capital stock (replacement cost in some base year gross or net of depreciation) in practice, but there are many problems (variations in quality, scrapping, using stock rather than flow).
  - Problems of quality
  - Problems of age/vintage
  - Conceptual problems: utilisation and price of capital $q = (r + \delta)$, rate of interest plus depreciation
  - Care needed that have not used identity $PQ = wL + mK$ to get capital

But in doing research you have to make assumptions and leaps of faith. Hopefully can justify them, or do sensitivity test for different assumptions. At least can be honest about any problems and take in interpreting results.

A lot of the empirical literature is about how problems are overcome when they are encountered.

Cross section studies:

Consider firm i with C-D production function

$$Q_i = AK_i^{\alpha}L_i^{\beta}\varepsilon_i$$

ε_i can be interpreted as measuring technical efficiency

- Problems:
- If firms are assumed to maximise profits and are price takers we can get the usual marginal productivity conditions and introduce random disturbances. The problem is that in cross section the prices the firms face are likely to be the same for all firms and so there is an identification problem. Production function could be confused with linear combination of the marginal productivity equations.
- Suppose data obeys the accounting identity $PQ = wL + mK$, which may have been used to calculate capital. Then estimating the production function would just rediscover the identity. McCombie and Dixon discuss this more generally.

Can overcome problems by:

- obtain data were prices do vary over the sample, imperfect competition, across countries.
- If disturbances are uncorrelated all equations are identified. But this is unlikely as technically efficient firms are likely to be economically efficient

But even if prices vary over the cross section there can still be problems of simultaneity bias.

Approaches to deal with problems:

- Kleins method: uses marginal productivity equations which are identified even when proces dont vary
- Nerlove (1963) looks at US electricity supply
- Thomas provides useful discussion.

Time series studies

When we move to time series data:

- Constancy of prices problem disappears, which reduces the identification problem
- but factor prices are ratios in the MP equations so problem if these change at the same rate over time
- Also technical progress over time leads to shifts in production function

For aggregate production functions:

- Usual conceptual problems of aggregation but product and factor prices do vary
- No identification problem though simultaneity bias and multicollinearity remain
- Again technical progress over time leads to shifts in production function

Solow and technical progress

Solow developed the C-D framework to deal with technical progress

$$Q_t = A(t)F(K_t, L_t)$$

assuming crt scale

$$\frac{Q_t}{L_t} = A(t)F(\frac{K_t}{L_t}, 1) = A(t)f(\frac{K_t}{L_t})$$

in per head variables

$$q_t = A(t)f(k_t)$$

changes in output per head

Example: Estimating a production function

Example is A Birdi and and JP Dunne “South Africa: An Econometric Analysis of Military Spending and Economic Growth”. Chapter 9 in Jurgen Brauer and J Paul Dunne (eds) (2002) "Arming the South: The Economics of Military Expenditures, Arms Production and Trade in Developing Countries". Palgrave.

Developments in production functions

There have been a range of developments in production fucntion estimation to generalise the function being estimated and to take into account some of the problems involved, such as inthe case of vintage effects. Thomas gives some info.

Variable elasticity of substitution

Problem of finding a functional form that allowed for variable elasticity of substitution but approximated the underlying production process

Christensen, Jorgensen and Lau

Transcedental logarithmic or translog production function

Frontier production functions

Aims to find the production possibility frontier and allow judgements to be made about

relative productivity of the categories under investigation -Universities, comapnies etc

Vintage models

Introduce the notion of capital being long lasting -not just issues of depreciaition, but also of adjustment to changes in expected output. Some capital is "putty", can be changed easily, some is "clay" and cannot.

Endogenous growth models

Birdi and Dunne (2002) have some discussion. Essentially this introduces increasing

returns to scale into aggregate production function, resulting from technology of human capital effects.